A new combinatorial characterization of the minimal cardinality of a subset of R which is not of first category

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Abstract

Let $\mathcal{M}$ denote the ideal of first category subsets of $\mathbf{R}$. We prove that $\min\{\text{card X} : \text{X} \subseteq \mathbf{R}, \text{X} \not\in \mathcal{M}\}$ is the smallest cardinality of a family $S \subseteq \{0,1\}^{\omega}$ with the property that for each $f : \omega \longrightarrow \bigcup_{n\in\omega}\{0,1\}^n$ there exists a sequence $\{a_n\}_{n\in\omega}$ belonging to S such that for infinitely many $i \in \omega$ the infinite sequence $\{a_{i+n}\}_{n\in\omega}$ extends the finite sequence $f(i)$.

We inform that $S \subseteq \{0,1\}^{\omega}$ is not of first category if and only if for each $f : \omega \longrightarrow \bigcup_{n\in\omega}\{0,1\}^n$ there exists a sequence $\{a_n\}_{n\in\omega}$ belonging to S such that for infinitely many $i \in \omega$ the infinite sequence $\{a_{i+n}\}_{n\in\omega}$ extends the finite sequence $f(i)$.

Let $\mathcal{M}$ denote the ideal of first category subsets of $\mathbf{R}$. Let $\mathcal{M}(\{0,1\}^{\omega})$ denote the ideal of first category subsets of the Cantor space $\{0,1\}^{\omega}$. Obviously:

$$(*)\qquad \begin{aligned} \text{non}(\mathcal{M}) :&= \min\{\text{card } X : X \subseteq \mathbf{R}, X \not\in \mathcal{M}\} \\ &= \min\{\text{card } X : X \subseteq \{0,1\}^{\omega}, X \not\in \mathcal{M}(\{0,1\}^{\omega})\} \end{aligned}$$

Let $\forall^{\infty}$ abbreviate "for all except finitely many". It is known (see [1], [2] and also [3]) that:

$$\text{non}(\mathcal{M}) =$$
$$\min\{\text{card } F : F \subseteq \omega^{\omega} \text{ and } \neg\ \exists g \in \omega^{\omega}\ \forall f \in F\ \forall^{\infty} k\ g(k) \neq f(k)\}$$

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For $S \subseteq \{0,1\}^\omega$ we define the following property $(**)$:

$(**)$ for each $f : \omega \to \bigcup_{n\in\omega}\{0,1\}^n$ there exists a sequence $\{a_n\}_{n\in\omega}$ belonging to S such that for infinitely many $i \in \omega$ the infinite sequence $\{a_{i+n}\}_{n\in\omega}$ extends the finite sequence $f(i)$.

Theorem 1 *If $S \subseteq \{0,1\}^\omega$ is not of first category then S has the property $(**)$.*

Proof. Let us fix $f : \omega \longrightarrow \bigcup_{n\in\omega}\{0,1\}^n$. Let $S_k(f)$ $(k \in \omega)$ denote the set of all sequences $\{a_n\}_{n\in\omega}$ belonging to $\{0,1\}^\omega$ with the property that there exists $i \geq k$ such that the infinite sequence $\{a_{i+n}\}_{n\in\omega}$ extends the finite sequence $f(i)$.

Sets $S_k(f)$ $(k \in \omega)$ are open and dense. In virtue of the Baire category theorem $\bigcap_{k\in\omega} S_k(f) \cap S$ is non-empty i.e. there exists a sequence $\{a_n\}_{n\in\omega}$ belonging to S such that for infinitely many $i \in \omega$ the infinite sequence $\{a_{i+n}\}_{n\in\omega}$ extends the finite sequence $f(i)$. This completes the proof.

Note. The author recently proved that if $S \subseteq \{0,1\}^\omega$ has the property $(**)$ then S is not of first category; the proof will appear in a separate preprint. From this result and Theorem 1 we obtain the following characterization:
$S \subseteq \{0,1\}^\omega$ is not of first category if and only if S has the property $(**)$.
Let us note that from the above characterization we may deduce all next results; therefore all next proofs are unnecessary.

Theorem 2 *If $S \subseteq \{0,1\}^\omega$ has the property $(**)$ then* card $S \geq$ non$(\mathcal{M})$.

Proof. For a sequence $\{a_n\}_{n\in\omega}$ belonging to S we define:

$$\tilde{a}_n := \sum_{i=n}^{\infty} \frac{a_i}{2^{i-n+1}} \in [0,1].$$

The following Observation is easy.

Observation. Assume that $S \subseteq \{0,1\}^\omega$ has the property $(**)$. We claim that for each sequence $\{U_k\}_{k\in\omega}$ of non-empty open sets satisfying $U_k \subseteq (0,1)$ $(k \in \omega)$ there exists a sequence $\{a_k\}_{k\in\omega}$ belonging to S such that for infinitely many $k \in \omega$ $\tilde{a}_k \in U_k$.

There exists a sequence $\{(c_i,d_i)\}_{i\in\omega}$ of non-empty pairwise disjoint intervals satisfying

$$\bigcup_{i\in\omega}(c_i,d_i) \subseteq (0,1).$$

We assign to each $\{a_n\} \in S$ the function $s_{\{a_n\}} : \omega \longrightarrow \omega$ according to the following rules (cf.[4]):

1) if $\tilde{a}_k \notin \bigcup_{i\in\omega}(c_i,d_i)$ then $s_{\{a_n\}}(k) = 0$,

2) if $\tilde{a}_k \in \bigcup_{i\in\omega}(c_i,d_i)$ then $s_{\{a_n\}}(k)$ is the unique $i \in \omega$ such that $\tilde{a}_k \in (c_i,d_i)$.

Suppose, contrary to our claim, that card $S <$ non$(\mathcal{M})$. It implies that the cardinality of the family $\{s_{\{a_n\}} : \{a_n\} \in S\} \subseteq \omega^\omega$ is also less than

$$\text{non}(M) =$$
$$\min\{\text{card } F : F \subseteq \omega^\omega \text{ and } \neg\, \exists\, g \in \omega^\omega\ \forall f \in F\ \forall^\infty k\ g(k) \neq f(k)\}$$

Therefore, there exists a function $g : \omega \longrightarrow \omega$ such that for each sequence $\{a_n\} \in S$ $\forall^\infty k$ $g(k) \neq s_{\{a_n\}}(k)$. We define $U_k := (c_{g(k)}, d_{g(k)}) \subseteq (0,1)$ $(k \in \omega)$. If $\{a_n\} \in S$ then the set $A_{\{a_n\}} := \{k \in \omega : g(k) = s_{\{a_n\}}(k)\}$ is finite and for each $k \in \omega \setminus A_{\{a_n\}}$ $\tilde{a}_k \notin U_k$. It contradicts the thesis of the Observation which ensures that there exists a sequence $\{a_k\}_{k\in\omega}$ belonging to S such that for infinitely many $k \in \omega$ $\tilde{a}_k \in U_k$. This completes the proof of Theorem 2.

Corollary. From $(*)$, Theorem 1 and Theorem 2 follows that $\mathrm{non}(\mathcal{M})$ is the smallest cardinality of a family $S \subseteq \{0,1\}^\omega$ with the property that for each $f : \omega \longrightarrow \bigcup_{n\in\omega}\{0,1\}^n$ there exists a sequence $\{a_n\}_{n\in\omega}$ belonging to S such that for infinitely many $i \in \omega$ the infinite sequence $\{a_{i+n}\}_{n\in\omega}$ extends the finite sequence $f(i)$.

Remark. Another (not purely combinatorial) characterizations of $\mathrm{non}(\mathcal{M})$ can be found in [4].

References

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